

Zak transform, Weil representation, and integral operators with theta-kernels

Foth T.¹, Neretin Yu.A.²

ABSTRACT. The Weil representation of a real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ admits a canonical extension to a holomorphic representation of a certain complex semigroup consisting of Lagrangian linear relations (this semigroup includes the Olshanski semigroup). We obtain the explicit realization of the Weil representation of this semigroup in the Cartier model, i.e., in the space of smooth sections of a certain line bundle on the $2n$ -dimensional torus \mathbb{T}^{2n} . We show that operators of the representation are integral operators whose kernels are theta-functions on \mathbb{T}^{4n} .

We also extend this construction to a functor from a certain category of Lagrangian linear relations between symplectic vector spaces of different dimensions to a category of integral operators acting on sections of line bundles on the tori.

1 Introduction

1.1. Theta-kernels. Let T be an $\alpha \times \alpha$ symmetric matrix with negative definite real part. Let z range in \mathbb{R}^α . We define a *theta-function* $\theta(T; z; \zeta)$ as a function in variables $(z, \zeta) \in \mathbb{R}^\alpha \times \mathbb{R}^\alpha$ given by

$$\theta[T; z; \zeta] = \sum_{k \in \mathbb{Z}^\alpha} \exp\left\{\frac{1}{2}(z + 2\pi k)^t T (z + 2\pi k) + ik \cdot \zeta\right\} \quad (1.1)$$

This expression is a variant of a multivariate theta-function, see [11], Chapter 2, Section 1 (see definition of theta-functions with characteristics).

Consider the space $L^2([0, 2\pi]^{2n})$. We write its elements as functions $f(x, \xi)$, where x and ξ are elements of $[0, 2\pi]^n$. We consider integral operators in $L^2([0, 2\pi]^{2n})$, with θ -kernels. More precisely, let $\alpha = 2n$, $z = \begin{pmatrix} x & y \end{pmatrix}^t \in \mathbb{R}^n \oplus \mathbb{R}^n$, $\zeta = (\xi, -\eta)^t \in \mathbb{R}^n \oplus \mathbb{R}^n$. Let $T = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ be a $(n+n) \times (n+n)$ matrix. We consider integral operators in $L^2([0, 2\pi]^{2n})$ having the form

$$Af(x, \xi) = \frac{1}{(2\pi)^{3n/2}} \int_{[0, 2\pi]^n \times [0, 2\pi]^n} \theta\left[\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}; x, y; \xi, \eta\right] f(y, \eta) dy d\eta \quad (1.2)$$

We show that these operators form a semigroup, i.e., the product of two such integral operators has the same form (see Proposition 3.5). In fact, multiplying two integral operators associated with matrices $\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ and $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ we

¹Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
foth@umich.edu

²ITEP, B.Chernushkinskaya 25, Moscow 117 259, Russia
neretin@mccme.ru

obtain the integral operator associated with the matrix

$$\begin{pmatrix} A - B(C + K)^{-1}B^t & -B(K + C)^{-1}L \\ -L^t(C + K)^{-1}B^t & M - L^t(C + K)^{-1}L \end{pmatrix}. \quad (1.3)$$

1.2. Lagrangian linear relations. The formula (1.3) for a strange multiplication of matrices admits a transparent algebraic interpretation. In fact, the semigroup of matrices with the multiplication (1.3) is isomorphic to a certain semigroup of Lagrangian linear relations (see its description below, in Section 2).

This semigroup of Lagrangian linear relations has several interesting realizations by integral operators.

First, it is isomorphic to the semigroup of integral operators in \mathbb{R}^n having the form

$$\mathcal{B}f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right\} f(y) dy \quad (1.4)$$

Equivalently, this semigroup is isomorphic to the semigroup of Gauss operators in the holomorphic (Segal–Bargmann) model of the boson Fock space with n degrees of freedom (see [14], [15], later this was obtained in [6]).

An infinite-dimensional variant of this semigroup has a realization in the space of symmetric functions, see [18].

All these realizations are different models of the Weil representation.³ A relation of the Weil representation with theta-functions seems well known, see [2], [10], [13]. A possibility to realize the Weil representation by integral operators with theta-kernels was conjectured by R. Howe (private discussion, 1994). Authors are grateful to R. Howe, A. A. Kirillov, and E. Kaniuth for discussions of the subject.

1.3. Structure of this paper. Our main result (Theorem 3.4) is the multiplication law for the operators (1.2) with θ -kernels. We also show (Theorem 3.3) that the Zak transform maps Gauss operators in $L^2(\mathbb{R}^n)$ to operators with θ -kernels in $L^2(\mathbb{T}^{2n})$. These theorems are proved in Section 3.

Section 2 contains preliminaries on Gauss operators (see also [7], [3], [16]–[17]) and Lagrangian linear relations (see also [16]).

2 Preliminaries. Gauss operators in $L^2(\mathbb{R}^n)$ and the category of Lagrangian linear relations.

2.1. Category of Gauss operators. Denote by $\mathcal{D}(\mathbb{R}^n)$ the space of complex-valued C^∞ -functions on \mathbb{R}^n with compact support. Denote by $\mathcal{S}(\mathbb{R}^n)$ the

³Other terms are the *Segal–Shale–Weil representation*, the *harmonic representation*, the *oscillator representation*. Apparently, it was discovered by K. O. Friedrichs (see [4]) near 1950. He formulates a correct theorem on the infinite dimensional symplectic group and proposes a proof “at the physical level”; it is not satisfactory from the mathematical point of view. But for finite dimensional symplectic groups his arguments are sufficient.

Schwartz space of smooth rapidly decreasing complex-valued functions on \mathbb{R}^n , denote by $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ their dual spaces, i.e., the space of all distributions on \mathbb{R}^n and the space of tempered distributions.

Fix integers $m, n \geq 0$. Denote by $\Omega_{m,n}$ the set of $(n+m) \times (n+m)$ block matrices of the form $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$, where

- 1) $S = S^t$, i.e., $K^t = K$, $M^t = M$,
- 2) $(-\operatorname{Re} S)$ is strictly positive definite.

For a matrix $S \in \Omega_{m,n}$, define the integral operator $L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^n)$ of the form

$$\mathcal{B}[S]f(x) = \frac{1}{(2\pi)^{(m+n)/4}} \int_{\mathbb{R}^m} \exp\left\{\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix}\right\} f(y) dy. \quad (2.1)$$

Since the kernel of the operator is square integrable, $\mathcal{B}[S]$ is a Hilbert-Schmidt operator $L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^n)$. Since the kernel is an element of $\mathcal{S}(\mathbb{R}^{n+m})$, our operator is a bounded operator $\mathcal{S}'(\mathbb{R}^m) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ (see a variant of the Kernel Theorem [5], Theorem 5.2.6).

It can be easily checked by a direct calculation (see Howe [7]⁴) that the product of two operators of this form has the same form, the precise statement is as follows.

Theorem 2.1 *Let*

$$S_1 = \begin{pmatrix} K_1 & L_1 \\ L_1^t & M_1 \end{pmatrix} \in \Omega_{n,k}, \quad S_2 = \begin{pmatrix} K_2 & L_2 \\ L_2^t & M_2 \end{pmatrix} \in \Omega_{m,n}$$

then the composition is given by

$$\mathcal{B}[S_1]\mathcal{B}[S_2] = \lambda(S_1, S_2)\mathcal{B}[S_3] \quad (2.2)$$

where

$$\lambda(S_1, S_2) = \det[(-M_1 - K_2)^{-1/2}] \quad (2.3)$$

and

$$S_3 = \begin{pmatrix} K_1 - L_1(M_1 + K_2)^{-1}L_1^t & -L_1(M_1 + K_2)^{-1}L_2 \\ -L_2^t(M_1 + K_2)^{-1}L_1^t & M_2 - L_2^t(M_1 + K_2)^{-1}L_2 \end{pmatrix}. \quad (2.4)$$

PROOF. To show that the formula (2.2) for the composition of the operators holds, it is sufficient to verify the identity

$$\begin{aligned} & \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} S_1 \begin{pmatrix} x \\ y \end{pmatrix}\right\} \exp\left\{\frac{1}{2} \begin{pmatrix} y^t & z^t \end{pmatrix} S_2 \begin{pmatrix} y \\ z \end{pmatrix}\right\} dy = \\ & = \lambda(S_1, S_2) \exp\left\{\frac{1}{2} \begin{pmatrix} x^t & z^t \end{pmatrix} S_3 \begin{pmatrix} x \\ z \end{pmatrix}\right\}. \end{aligned} \quad (2.5)$$

⁴To be precise, he considers the case $m = n$

But the integral in the left hand side is the usual Gauss integral of the form

$$\int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}y^t A y + y^t b\right\} dy = \frac{(2\pi)^{n/2}}{\det A^{1/2}} \exp\left\{-\frac{1}{2}b^t A^{-1} b\right\}$$

REMARK. For a symmetric complex $n \times n$ matrix T satisfying $\operatorname{Re} T > 0$, the determinat $\det(T^{-1/2})$ is well defined. Indeed, for each $v \in \mathbb{C}^n$, we have $\operatorname{Re}\langle Tv, v \rangle > 0$. Let v be an eigenvector of T with the eigenvalue λ . Then $0 < \operatorname{Re}\langle Tv, v \rangle = \operatorname{Re} \lambda \langle v, v \rangle$ and hence $\operatorname{Re} \lambda_j > 0$. Therefore we can assume $\det(T^{-1/2}) = \prod \lambda_j^{-1/2}$. \square

Thus we obtain some category \mathcal{K} , whose objects are $0, 1, 2, \dots$, set of morphisms $m \rightarrow n$ is $\Omega_{m,n}$, and the product of morphisms is given by formula (2.4).

In particular, the set $\Omega_{n,n}$ is a semigroup. This semigroup was discussed in the work of Howe [7] and the work of Olshanski [19] that was unpublished for a long time. In particular, it was observed that the set of matrices with the invertible block L is isomorphic to a subsemigroup in $\operatorname{Sp}(2n, \mathbb{C})$ (see Section 2.4). Nevertheless, the semigroup $\Omega_{n,n}$ itself can not be embedded to any group (since any matrix of the form $\begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}$ is an idempotent).

It appears [14], [15] that the multiplication (2.4) hides quite simple algebraic structure, it is present in next subsection.

REMARK. It is natural to consider slightly more general operators, whose kernels have the form

$$\exp\left\{\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix}\right\} \delta_L(x, y) \quad (2.6)$$

where S is a *nonpositive* definite matrix, and δ_L is a δ -function of some linear subspace $L \subset \mathbb{R}^m \oplus \mathbb{R}^n$. Obviously, each kernel of this form is a limit of kernels of the form (2.10).

We restrict ourself to the case described above. \square

2.2. Category of Lagrangian linear relations

Let V, W be linear spaces. A *linear relation* $P : V \rightrightarrows W$ is a linear subspace in $V \oplus W$. If $P : V \rightrightarrows W, Q : W \rightrightarrows Y$ are linear relations, then their product QP is the linear relation $V \rightrightarrows Y$ consisting of vectors $(v, y) \in V \oplus Y$ satisfying the condition: there exists $w \in W$ such that $(v, w) \in P, (w, y) \in Q$.

EXAMPLE. Let $A : V \rightarrow W$ be a linear operator. Then its graph $\operatorname{graph}(A)$ is a linear relation. If $A : V \rightarrow W, B : W \rightarrow Y$ are linear operators, then

$$\operatorname{graph}(BA) = \operatorname{graph}(B)\operatorname{graph}(A) \quad \square$$

We shall need the definition of the *rank* of a linear relation

$$\operatorname{rk} P := \dim P - \dim(P \cap V) - \dim(P \cap W)$$

If P is a graph of a linear operator A , then $\operatorname{rk} P = \operatorname{rk} A$.

Consider the space

$$V_n := V_n^+ \oplus V_n^- = \mathbb{C}^n \oplus \mathbb{C}^n$$

We equip this space with two forms, a skew symmetric bilinear form L_n

$$L_n((v_+, v_-); (w_+, w_-)) := \sum_{j=1}^n (v_+^j w_-^j - v_-^j w_+^j)$$

and a Hermitian form H_n

$$H_n((v_+, v_-); (w_+, w_-)) := \frac{1}{i} \sum_{j=1}^n (v_+^j \bar{w}_-^j - v_-^j \bar{w}_+^j)$$

REMARK. Let g be a *real* matrix preserving the skew-symmetric form L_n . Obviously, this matrix also preserves the form H_n . Conversely, let g preserves the both forms L_n, H_n . Then it commutes with complex conjugation and hence g is a real matrix. Thus, the group of linear operators, preserving these two forms is the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. \square

REMARK. Let us explain how the space V_n with two forms appears in a natural way. Consider a space \mathbb{R}^{2n} equipped with a skew-symmetric bilinear form M . Consider its complexification \mathbb{C}^{2n} . We can extend M to \mathbb{C}^{2n} as a bilinear form, or as a sesquilinear form. This gives two forms as above. \square

For each m, n , we define two forms on $V_{2m} \oplus V_{2n}$.

$$L_{m,n}((v, w), (v', w')) = L_m(v, v') - L_n(w, w')$$

$$H_{m,n}((v, w), (v', w')) = H_m(v, v') - H_n(w, w')$$

We define⁵ a category **Sp** whose objects are the spaces V_n . We define a morphism V_n to V_m as a linear relation $P : V_m \rightrightarrows V_n$ such that

1. P is Lagrangian⁶ with respect to the form $L_{m,n}$. In particular $\dim P = m + n$.

2. the form $H_{m,n}$ is strictly positive definite on P .

We denote the set of all morphisms $m \rightarrow n$ by $\mathrm{Mor}(V_m, V_n)$ or $\mathrm{Mor}_{\mathbf{Sp}}(V_m, V_n)$.

Proposition 2.2 *Let P be a morphism V_m to V_n , and Q be a morphism V_n to V_k . Then QP is a morphism V_m to V_k .*

For proof, see [17], V.1.

In particular, the set $\mathrm{End}(V_n)$ of morphisms V_n to V_n forms a semigroup.

As we will observe below in Section 2.3, the category **Sp** is equivalent to the category \mathcal{K} defined in the previous subsection.

⁵As far as the authors know, this category firstly appeared in [9].

⁶This means that the skew-symmetric form is zero on P and P has the maximal possible dimension, in our case $\dim P = m + n$.

REMARK. Let $g \in \text{Sp}(2n, \mathbb{R})$. Then $\text{graph}(g)$ is Lagrangian with respect to the form L_n and the form H_n is zero on $\text{graph}(g_n)$. Thus, the group $\text{Sp}(2n, \mathbb{R})$ lies on the boundary of the semigroup $\text{Mor}(V_n, V_n)$. A variant of the definition of the category **Sp** including the groups $\text{Sp}(2n, \mathbb{R})$ is discussed in [17]. \square

2.3. Coordinates on $\text{Mor}(V_m, V_n)$.

Lemma 2.3 a) Let $P \in \text{Mor}(V_m, V_n)$. Then there exists an $(n+m) \times (n+m)$ matrix (V.P.Potapov transform) $S = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ such that P is the space of solutions of the following system of linear equations

$$\begin{pmatrix} -v^+ \\ w^+ \end{pmatrix} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} v^- \\ w^- \end{pmatrix}$$

where

$$v^\pm \in V_n^\pm; \quad w^\pm \in V_m^\pm$$

Moreover, S satisfies the conditions

1. S is symmetric ($S = S^t$)
2. $\text{Re } S < 0$
- b) The map $P \mapsto S(P)$ is a bijection of the set $\text{Mor}(V_m, V_n)$ to the set of $(n+m) \times (n+m)$ matrices satisfying conditions 1-2.
- c) The product in these coordinates is given by the following formula: if $P \in \text{Mor}(V_m, V_n)$, $Q \in \text{Mor}(V_k, V_m)$, and

$$S(P) = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \quad S(Q) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$$

are their Potapov transforms, then

$$S(PQ) = \begin{pmatrix} A - B(C + K)^{-1}B^t & -B(K + C)^{-1}L \\ -L^t(C + K)^{-1}B^t & M - L^t(C + K)^{-1}L \end{pmatrix}. \quad (2.7)$$

PROOF. a)- b). The Hermitian form $H_{m,n}$ is zero on $V_n^+ \oplus V_m^+$ and it is strictly positive on P . Hence $P \cap (V_n^+ \oplus V_m^+) = 0$. Since $\dim P = m + n = \dim V_n^- \oplus V_m^-$, the subspace P is the graph of an operator $V_n^- \oplus V_m^- \rightarrow V_n^+ \oplus V_m^+$.

Our subspace P is Lagrangian with respect to the skew-symmetric form $L_{m,n}$. This is equivalent to $S = S^t$.

The positivity of the Hermitian form $H_{m,n}$ is equivalent to the condition $\text{Re } S < 0$.

c) We have the system of equations

$$\begin{aligned} v^+ &= -Av^- - Bw^-; & w^+ &= -Kw^- - Ly^-; \\ w^+ &= B^tv^- + Cw^-; & y^+ &= L^tw^- + My^- \end{aligned}$$

where

$$v^\pm \in V_n^\pm; \quad w^\pm \in V_m^\pm; \quad y^\pm \in V_l^\pm$$

Subtracting the second equation from the third one, we obtain

$$w^- = -(C + K)^{-1}(B^t v_- + Ly_-)$$

Substituting the expression for w^- to the first and the last equations, we get

$$v^+ = -(A - B(C + K)^{-1}B^t)v^- + B(K + C)^{-1}Ly_-$$

$$w^- = -L^t(C + K)^{-1}B^t v^- + (M - L^t(C + K)^{-1}L)y_-.$$

as it was required. \square

2.4. Remark: Linear operators that are contained in $\text{End}(V_n)$.

If $n \neq m$, then $P \in \text{Mor}(V_n, V_m)$ is not a graph of an operator $V_n \rightarrow V_m$ (since $\dim P = m + n$).

Let $m = n$. Consider the semigroup Γ_n (*Olshanski semigroup* [20]) consisting of operators $g : V_n \rightarrow V_n$ such that

- a) g preserves the skew-symmetric bilinear form L_n , i.e., $g \in \text{Sp}(2n, \mathbb{C})$.
- b) For each nonzero $v \in V_n$,

$$H_n(gv, gv) < H_n(v, v)$$

If $g \in \Gamma_n$, then $\text{graph}(g) \in \text{Mor}(V_n, V_n)$.

The semigroup Γ_n is open (nondense) in $\text{Sp}(2n, \mathbb{C})$, the subgroup $\text{Sp}(2n, \mathbb{R}) \subset \text{Sp}(2n, \mathbb{C})$ is contained in the closure of Γ_n . The semigroup Γ_n is open and dense in $\text{End}(V_n)$.

An element P of $\text{End}(V_n)$ is contained in Γ_n iff B is invertible.

Let $g = \begin{pmatrix} P & Q \\ R & T \end{pmatrix} \in \Gamma_n$, i.e.,

$$\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \begin{pmatrix} P & Q \\ R & T \end{pmatrix} \begin{pmatrix} w^+ \\ w^- \end{pmatrix}$$

Expressing v^+ , w^+ in terms of v^- , w^- , we obtain that the Potapov transform of g is

$$S = \begin{pmatrix} -PR^{-1} & -Q + PR^{-1}T \\ R^{-1} & -R^{-1}T \end{pmatrix}$$

2.5. Explicit correspondence between Lagrangian linear relations and Gauss operators. Let $P \in \text{Mor}(V_n, V_m)$. Let

$$S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$$

be its Potapov transform. Consider the integral operator

$$\mathcal{B}[S(P)] : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^m)$$

given by

$$(\mathcal{B}[S(P)]f)(x) = \frac{1}{(2\pi)^{(n+m)/4}} \int_{\mathbb{R}^n} \exp\left\{\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} S(P) \begin{pmatrix} x \\ y \end{pmatrix}\right\} f(y) dy.$$

Theorem 2.4 Let $P \in \text{Mor}(V_n, V_m)$, $Q \in \text{Mor}(V_m, V_l)$, and $S(P) = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$, $S(Q) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be their Potapov transforms. Then

$$\mathcal{B}[S(Q)]\mathcal{B}[S(P)] = \lambda(Q, P)\mathcal{B}[S(QP)]$$

where

$$\lambda(P, Q) = \det(-C - K)^{-1/2}.$$

This theorem is a corollary of the multiplication formula (2.7) and formulae (2.2)-(2.4) for the product of integral operators.

REMARK We can say that $P \mapsto \mathcal{B}[S(P)]$ is a projective representation of the category \mathbf{Sp} , on definitions of representations of categories see [17].

REMARK. As we mention above, in [17] another definition of category \mathbf{Sp} , which gives a slightly larger structure, is used. This category is equivalent to enlarged category of Gauss operators mentioned in 2.1. Explicit formulae for the correspondence between elements of $\text{End}(V_n)$ and Gauss operators are contained in [6].

2.6. Heisenberg algebra and another description of the correspondence between linear relations and Gauss operators

To each

$$\alpha = (\alpha^+; \alpha^-)^t = (\alpha_1^+, \dots, \alpha_n^+; \alpha_1^-, \dots, \alpha_n^-)^t \in V_n = V_n^+ \oplus V_n^-$$

we associate an operator

$$\hat{A}(\alpha) = \sum \alpha_j^+ x_j + \sum \alpha_j^- \frac{\partial}{\partial x_j}.$$

All operators of this type form the complex Heisenberg algebra.

Proposition 2.5 Let $P \in \text{Mor}(V_n, V_m)$.

a) For each $(\alpha, \beta) \in P$,

$$\hat{A}(\alpha)\mathcal{B}[S(P)] = \mathcal{B}[S(P)]\hat{A}(\beta)$$

b) Let $R : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^m)$ be a bounded operator satisfying the equality

$$\hat{A}(\alpha)R = R\hat{A}(\beta)$$

for all $(\alpha, \beta) \in P$. Then $R = \lambda\mathcal{B}[S(P)]$ for some $\lambda \in \mathbb{C}$.

PROOF. a) It is sufficient to prove that the kernel $K = K(x, y)$ of $\mathcal{B}[P]$ satisfies

$$\left(\sum \alpha_j^+ x_j + \sum \alpha_j^- \frac{\partial}{\partial x_j} \right) \int_{\mathbb{R}^n} K(x, y) f(y) dy =$$

$$= \int_{\mathbb{R}^n} K(x, y) \left(\sum \beta_k^+ y_k + \sum \beta_k^- \frac{\partial}{\partial y_k} \right) f(y) dy$$

This is equivalent to the following system of differential equations:

$$\begin{aligned} & \sum_{j=1}^m (\alpha_j^+ x_j + \alpha_j^- \frac{\partial}{\partial x_j}) \exp \left\{ \frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\} = \\ & = \sum_{k=1}^n (\beta_k^+ y_k - \beta_k^- \frac{\partial}{\partial y_k}) \exp \left\{ \frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\} \end{aligned}$$

After differentiation, we obtain

$$\alpha^+ = -A\alpha^- - B\beta^-; \quad \beta^+ = B^t\alpha^- + C\beta^- \quad (2.8)$$

b) By a Kernel Theorem (see [5], 5.2.6) the operator R has the form

$$Rf(x) = \int_{\mathbb{R}^n} L(x, y) f(y) dy$$

where $L \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$. The distribution L must satisfy

$$\sum_{j=1}^m (\alpha_j^+ x_j + \alpha_j^- \frac{\partial}{\partial x_j}) L(x, y) = \sum_{k=1}^n (\beta_k^+ y_k - \beta_k^- \frac{\partial}{\partial y_k}) L(x, y) \quad (2.9)$$

for all α, β .

Consider the distribution

$$M(x, y) := L(x, y) \exp \left\{ -\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

The system (2.9) implies

$$\sum_{j=1}^m (\alpha_j^- \frac{\partial}{\partial x_j}) M(x, y) = \sum_{k=1}^n (\beta_k^- \frac{\partial}{\partial y_k}) M(x, y)$$

for all α_j^-, β_k^- . Hence $M(x, y)$ is a constant. \square

REMARK. Let $n = 0$. Let $P \in \text{Mor}_{\mathbf{Sp}}(V_0, V_n)$. Its Potapov transform is $(m+0) \times (m+0)$ matrix A . The corresponding Gauss operator

$$\mathcal{B}(P) : L^2(\mathbb{R}^0) = \mathbb{R} \rightarrow L^2(\mathbb{R}^m)$$

is given by

$$s \mapsto s \cdot \exp \left\{ \frac{1}{2} x^t A x \right\}$$

3 Integral operators with theta-kernels

3.1. Zak transform and its properties

Consider tori $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$, $\mathbb{T}^{2n} = \mathbb{R}^{2n} / (2\pi\mathbb{Z})^{2n}$.

Denote by $G(\mathbb{R}^{2n})$ the subspace of $C^\infty(\mathbb{R}^{2n})$ that consists of functions $g(x, \xi)$ with the properties

$$g(x, \xi + 2\pi k) = g(x, \xi), \quad g(x + 2\pi k, \xi) = e^{-ik \cdot \xi} g(x, \xi) \quad (3.1)$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$. We equip this space with the topology of uniform convergence with all derivatives on $[0, 2\pi]^n$.

We also define an inner product in $G(\mathbb{R}^{2n})$ by the formula

$$\langle g_1, g_2 \rangle = \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^{2n}} g_1(x, \xi) \overline{g_2(x, \xi)} dx d\xi$$

REMARK. In this formula we can replace integration over the cube $[0, 2\pi]^{2n}$ by integration over an arbitrary fundamental domain of the lattice \mathbb{Z}^{2n} in \mathbb{R}^{2n} . \square

The completion of $G(\mathbb{R}^{2n})$ with respect to this inner product is identified with L^2 on the cube $[0, 2\pi]^{2n}$.

REMARK. Obviously, we can naturally identify the space $G(\mathbb{R}^{2n})$ with the space of smooth sections of a certain line bundle $\mathcal{L} \rightarrow \mathbb{T}^{2n}$. Indeed, $\mathcal{L} = \mathbb{R}^{2n} \times \mathbb{C} / \sim$, where the equivalence relation is $(x, \xi, \zeta) \sim (x + 2\pi k, \xi + 2\pi m, e^{ik \cdot \xi} \zeta)$ for any $x, \xi \in \mathbb{R}^n$, $m, k \in \mathbb{Z}^n$, $\zeta \in \mathbb{C}$. \square

For a function f on \mathbb{R}^n we define its *Zak transform*⁷ $\mathfrak{Z}_n f$ by the formula

$$\mathfrak{Z}_n : f(x) \mapsto g(x, \xi) = \sum_{k \in \mathbb{Z}^n} f(x + 2\pi k) e^{ik \cdot \xi}.$$

where $\xi \in \mathbb{R}^n$. It is easy to see that the function $g(x, \xi)$ satisfies (3.1).

Theorem 3.1 (see [8], **Problems 475, 666, [3]**) a) *The Zak transform is a bounded invertible operator*

$$\mathfrak{Z}_n : \mathcal{S}(\mathbb{R}^n) \rightarrow G(\mathbb{R}^{2n}) \quad (3.2)$$

b) *The inverse transform is given by*

$$(\mathfrak{Z}_n^{-1} g)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g(x, \xi) d\xi.$$

c) *The Zak transform is a unitary operator $L^2(\mathbb{R}^n) \rightarrow L^2([0, 2\pi]^{2n})$.*

REMARK. The Zak transform has the following property unusual for the classical theory of integral transforms: it changes a functional dimension; i.e., it identifies a space of functions of n variables and a space of functions of $2n$ variables.

⁷Another term is also used: *Weil–Brezin transform*, see [1].

Proposition 3.2 ([2]) *The Zak transform maps the operator $f \mapsto x_j f$ in $L^2(\mathbb{R}^n)$ to the operator $g(x, \xi) \mapsto (x_j + \frac{2\pi}{i} \frac{\partial}{\partial \xi_j})$ and the operator $\partial/\partial x_j$ to $\partial/\partial x_j$.*

REMARK. In other words, the Zak transform intertwines the standard representation of the Heisenberg algebra in $\mathcal{S}(\mathbb{R})$ given by the operators $x_j, \partial/\partial x_j$ and the representation in $G(\mathbb{R}^{2n})$ given by the operators $\frac{1}{i} \frac{\partial}{\partial x_j}$ and $x_j + \frac{2\pi}{i} \frac{\partial}{\partial \xi_j}$, $j = 1, \dots, n$, see also [10], [13], Section 2. \square

3.2. θ -functions, see also [13], Section 2. For a negative definite $a \times a$ symmetric matrix T define a theta-function

$$\theta[T; z, \zeta] = \sum_{k \in \mathbb{Z}^a} \exp\left\{\frac{1}{2}(z + 2\pi k)^t T (z + 2\pi k)\right\} e^{ik \cdot \zeta},$$

$z, \zeta \in \mathbb{R}^a$, clearly $\theta[T; z, \zeta]$ is the image $\mathfrak{Z}_a f$ of the Gaussian $f(z) = \exp\{\frac{1}{2} z^t T z\}$ under the Zak transform.

By unitarity of the Zak transform,

$$\langle \theta[T_1; z, \zeta], \theta[T_2; z, \zeta] \rangle = \int_{\mathbb{R}^a} \exp\left\{\frac{1}{2} z^t (T_1 + T_2) z\right\} dz = \frac{(2\pi)^{\frac{a}{2}}}{\det(-T_1 - T_2)^{-1/2}}. \quad (3.3)$$

(below we use a modified variant of this identity in Section 3.5).

The Gaussian $\exp\{\frac{1}{2} x^t T x\}$ satisfies the differential equation

$$\sum_{j=1}^a (\gamma_j^+ z_j + \gamma_j^- \frac{\partial}{\partial z_j}) \exp\left\{\frac{1}{2} z^t T z\right\} = 0 \quad \text{if } \gamma^+ = -T \gamma^-.$$

By Proposition 3.2 the corresponding θ -function satisfies the equation

$$\sum_{j=1}^a (\gamma_j^+ (z_j + \frac{2\pi}{i} \frac{\partial}{\partial \zeta_j}) + \gamma_j^- \frac{\partial}{\partial z_j}) \theta[T; z, \zeta] = 0. \quad (3.4)$$

3.3. Formulation of result

For a symmetric $(m+n) \times (m+n)$ matrix S denote $b(x, y) = \frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix}$, $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, define a theta-kernel $\mathcal{K}_{n,m}[S](x, \xi; y, \eta)$ as

$$\mathcal{K}_{n,m}[S](x, \xi; y, \eta) = \sum_{k \in \mathbb{Z}^m} \sum_{l \in \mathbb{Z}^n} \exp\{b(x + 2\pi k, y + 2\pi l)\} e^{ik \cdot \xi} e^{-il \cdot \eta},$$

and the corresponding integral operator

$$\mathcal{Q}_{n,m}[S] : G(\mathbb{R}^{2n}) \rightarrow G(\mathbb{R}^{2m})$$

by

$$(\mathcal{Q}_{n,m}[S]g)(x, \xi) = \frac{1}{(2\pi)^{3(n+m)/4}} \int_{\mathbb{T}^{2n}} \mathcal{K}_{n,m}[S](x, \xi; y, \eta) g(y, \eta) dy d\eta.$$

where $g \in G(\mathbb{R}^{2n})$.

Theorem 3.3 *Let $P_1 \in \text{Mor}_{\mathbf{Sp}}(V_q, V_m)$, $P_2 \in \text{Mor}_{\mathbf{Sp}}(V_n, V_q)$ be linear relations. Let*

$$S(P_1) = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \quad S(P_2) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \quad (3.5)$$

be their Potapov transforms. Then the following formula holds for the composition

$$\mathcal{Q}_{q,m}[S(P_1)]\mathcal{Q}_{n,q}[S(P_2)] = \det(-C - K)^{-1/2}\mathcal{Q}_{n,m}[S(P_1P_2)]. \quad (3.6)$$

In particular, we obtain a projective representation of the category \mathbf{Sp} .

Theorem 3.4 $\mathfrak{Z}_m B[S]\mathfrak{Z}_n^{-1} = \mathcal{Q}_{n,m}[S]$

The rest of the paper is the proof of these theorems.

3.4. Computation of the kernel. Proof of Theorem 3.4

Now we shall prove that the operator

$$Q : \mathfrak{Z}_m \mathcal{B}[S]\mathfrak{Z}_n^{-1} : G(\mathbb{R}^{2n}) \rightarrow G(\mathbb{R}^{2m})$$

is equal up to a scalar factor to the operator with the kernel $\frac{1}{(2\pi)^{3(n+m)/4}}\mathcal{K}_{n,m}[S]$, where

$$\mathcal{K}_{n,m}[S](x, \xi; y, \eta) = \sum_{k \in \mathbb{Z}^m} \sum_{l \in \mathbb{Z}^n} K(x + 2\pi k, y + 2\pi l) e^{ik \cdot \xi} e^{-il \cdot \eta},$$

where $K(x, y) = \exp\{\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix}\}$, i.e. $\frac{1}{(2\pi)^{(n+m)/4}}K(x, y)$ is the kernel of $\mathcal{B}[S]$. In Section 3.6 we will show that the scalar factor is 1.

First, \mathcal{K} satisfies the quasiperiodicity conditions

$$\begin{aligned} \mathcal{K}_{n,m}[S](x + 2\pi k, \xi; y + 2\pi l, \eta) &= e^{-i\xi \cdot k + i\eta \cdot l} \mathcal{K}_{n,m}[S](x, \xi; y, \eta), \\ \mathcal{K}_{n,m}[S](x, \xi + 2\pi k; y, \eta + 2\pi l) &= \mathcal{K}_{n,m}[S](x, \xi; y, \eta). \end{aligned}$$

and hence \mathcal{K} is the kernel of an operator $G(\mathbb{R}^n) \rightarrow G(\mathbb{R}^m)$.

Recall that for $(\alpha, \beta) \in P$, or equivalently, for α, β satisfying (2.8)

$$\hat{A}(\alpha)\mathcal{B}[S] = \mathcal{B}[S]\hat{A}(\beta), \quad (3.7)$$

where

$$\hat{A}(\alpha) = \sum_{j=1}^m (\alpha_j^+ x_j + \alpha_j^- \frac{\partial}{\partial x_j}), \quad \hat{A}(\beta) = \sum_{j=1}^n (\beta_j^+ y_j + \beta_j^- \frac{\partial}{\partial y_j}),$$

and we showed that $K(x, y)$ satisfies the differential equation

$$\sum_{j=1}^m (\alpha_j^+ x_j + \alpha_j^- \frac{\partial}{\partial x_j}) K(x, y) = \sum_{j=1}^n (\beta_j^+ y_j - \beta_j^- \frac{\partial}{\partial y_j}) K(x, y). \quad (3.8)$$

Applying the Zak transform to (3.7) we get:

$$(\mathfrak{Z}_m \hat{A}(\alpha) \mathfrak{Z}_m^{-1} Qg)(x, \xi) = (Q \mathfrak{Z}_n \hat{A}(\beta) \mathfrak{Z}_n^{-1} g)(x, \xi),$$

where $g \in G(\mathbb{R}^{2n})$. Hence the kernel \mathcal{K} of the operator Q satisfies the identity

$$\begin{aligned} \mathfrak{Z}_m \hat{A}(\alpha) \mathfrak{Z}_m^{-1} \int_{[0, 2\pi]^{2n}} \tilde{\mathcal{K}}(x, \xi; y, \eta) g(y, \eta) dy d\eta = \\ \int_{[0, 2\pi]^{2n}} \tilde{\mathcal{K}}(x, \xi; y, \eta) \mathfrak{Z}_n \hat{A}(\beta) \mathfrak{Z}_n^{-1} g(y, \eta) dy d\eta. \end{aligned}$$

By Proposition 3.2

$$\begin{aligned} \mathfrak{Z}_m \hat{A}(\alpha) \mathfrak{Z}_m^{-1} &= \sum_{j=1}^m (\alpha_j^+ (x_j + \frac{2\pi}{i} \frac{\partial}{\partial \xi_j}) + \alpha_j^- \frac{\partial}{\partial x_j}), \\ \mathfrak{Z}_n \hat{A}(\beta) \mathfrak{Z}_n^{-1} &= \sum_{j=1}^n (\beta_j^+ (y_j + \frac{2\pi}{i} \frac{\partial}{\partial \eta_j}) + \beta_j^- \frac{\partial}{\partial y_j}). \end{aligned}$$

Hence $\tilde{\mathcal{K}}(x, \xi; y, \eta)$ satisfies the differential equations

$$\begin{aligned} \sum_{j=1}^m (\alpha_j^+ (x_j + \frac{2\pi}{i} \frac{\partial}{\partial \xi_j}) + \alpha_j^- \frac{\partial}{\partial x_j}) \tilde{\mathcal{K}}(x, \xi; y, \eta) = \\ = \sum_{j=1}^n (\beta_j^+ (y_j + \frac{2\pi}{i} \frac{\partial}{\partial \eta_j}) + \beta_j^- \frac{\partial}{\partial y_j}) \tilde{\mathcal{K}}(x, \xi; y, \eta) \end{aligned}$$

and if $\tilde{\mathcal{K}} = \text{const} \cdot \mathcal{K}_{n,m}[S]$ then they are satisfied because of (3.4) with $a = n+m$,

$$\gamma^+ = \begin{pmatrix} \alpha^+ \\ -\beta^+ \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} \alpha^- \\ \beta^- \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \zeta = \begin{pmatrix} \xi \\ -\eta \end{pmatrix}, \quad T = S.$$

Note that the condition $\gamma^+ = -T\gamma^-$ is satisfied by (2.8).

Finally we note that the operator $\mathcal{B}[S] : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^m)$ satisfying (3.7) is unique up to a constant factor by Proposition 2.5 (b), therefore, clearly, the corresponding operator $Q : G(\mathbb{R}^{2n}) \rightarrow G(\mathbb{R}^{2m})$ is unique up to a scalar factor too.

3.5. Composition formula. Proof of Theorem 3.3.

Let $Q_1 := \mathcal{Q}_{q,m}[S_1]$ and $Q_2 := \mathcal{Q}_{n,q}[S_2]$ be two operators with the kernels $\frac{1}{(2\pi)^{3(q+m)/4}} \mathcal{K}_1$ and $\frac{1}{(2\pi)^{3(n+q)/4}} \mathcal{K}_2$ respectively, where

$$\mathcal{K}_1 = \mathcal{K}_1(x, \xi; y, \eta) = \mathcal{K}_{q,m}[S_1](x, \xi; y, \eta) = \sum_{k \in \mathbb{Z}^m} \sum_{l \in \mathbb{Z}^q} e^{b_1(x+2\pi k, y+2\pi l)} e^{ik \cdot \xi} e^{-il \cdot \eta},$$

$$\mathcal{K}_2 = \mathcal{K}_2(x, \xi; y, \eta) = \mathcal{K}_{n,q}[S_2](x, \xi; y, \eta) = \sum_{k \in \mathbb{Z}^q} \sum_{l \in \mathbb{Z}^n} e^{b_2(x+2\pi k, y+2\pi l)} e^{ik \cdot \xi} e^{-il \cdot \eta},$$

and $b_1(.,.)$ and $b_2(.,.)$ are the quadratic forms associated to S_1, S_2 .

Our Theorem 3.3 is equivalent to

Proposition 3.5 *The composition $Q_3 = Q_1 Q_2$ is the operator defined by*

$$(Q_3 g)(x, \xi) = \frac{1}{(2\pi)^{3(n+m)/4}} \int_{[0, 2\pi]^{2n}} \mathcal{K}_3(x, \xi; y, \eta) g(y, \eta) dy d\eta,$$

where $\mathcal{K}_3(x, \xi; y, \eta)$ is given by

$$\lambda(S_1, S_2) \sum_{k \in \mathbb{Z}^m} \sum_{l \in \mathbb{Z}^n} \exp\left\{\frac{1}{2} \begin{pmatrix} x^t + 2\pi k^t & y^t + 2\pi l^t \end{pmatrix} S_3 \begin{pmatrix} x + 2\pi k \\ y + 2\pi l \end{pmatrix}\right\} e^{ik \cdot \xi} e^{-il \cdot \eta}, \quad (3.9)$$

where $\lambda(S_1, S_2)$ is given by (2.3) and S_3 is given by (2.4).

Proof. We have:

$$(Q_3 g)(x, \xi) = (Q_1 Q_2 g)(x, \xi) = \frac{1}{(2\pi)^{3(n+m+2q)/4}} \int_{[0, 2\pi]^{2q}} \int_{[0, 2\pi]^{2n}} \mathcal{K}_1(x, \xi; s, \zeta) \mathcal{K}_2(s, \zeta; y, \eta) g(y, \eta) dy d\eta ds d\zeta,$$

hence

$$\begin{aligned} (2\pi)^{3q/2} \mathcal{K}_3(x, \xi; y, \eta) &= \int_{[0, 2\pi]^{2q}} \mathcal{K}_1(x, \xi; s, \zeta) \mathcal{K}_2(s, \zeta; y, \eta) ds d\zeta = \\ &= \int_{[0, 2\pi]^{2q}} \sum_{k \in \mathbb{Z}^m} \sum_{p \in \mathbb{Z}^q} e^{b_1(x+2\pi k, s+2\pi p)} e^{ik \cdot \xi} e^{-ip \cdot \zeta} \sum_{r \in \mathbb{Z}^q} \sum_{l \in \mathbb{Z}^n} e^{b_2(s+2\pi r, y+2\pi l)} e^{ir \cdot \zeta} e^{-il \cdot \eta} ds d\zeta = \\ &= \sum_k \sum_l e^{ik \cdot \xi} e^{-il \cdot \eta} \int_{[0, 2\pi]^{2q}} \sum_p \sum_r e^{b_1(x+2\pi k, s+2\pi p)} e^{b_2(s+2\pi r, y+2\pi l)} e^{ir \cdot \zeta} e^{-ip \cdot \zeta} ds d\zeta. \end{aligned}$$

For fixed x, y, ξ, η the integral above is $(2\pi)^q$ times the inner product (in $G(\mathbb{R}^{2q})$) of the Zak transforms of two Gaussian functions, therefore the expression above becomes

$$\frac{1}{(2\pi)^{q/2}} \sum_k \sum_l e^{ik \cdot \xi} e^{-il \cdot \eta} \left\langle \mathfrak{Z}_q(e^{b_1(x+2\pi k, s)}), \mathfrak{Z}_q(e^{b_2(s, y+2\pi l)}) \right\rangle_{L^2([0, 2\pi]^{2q})}.$$

The Zak transform (3.2) is unitary. Hence we can replace the inner product in $L^2[0, 2\pi]^{2q}$ by the inner product in $L^2(\mathbb{R}^q)$. Therefore we obtain:

$$\mathcal{K}_3(x, \xi; y, \eta) = \frac{1}{(2\pi)^{q/2}} \sum_k \sum_l e^{ik \cdot \xi} e^{-il \cdot \eta} \int_{\mathbb{R}^q} e^{b_1(x+2\pi k, s)} e^{b_2(s, y+2\pi l)} ds.$$

The last integral is the integral (2.5) with x, y, z, n replaced by $x + 2\pi k, s, y + 2\pi l, q$, and this finishes the proof.

3.6. End of proof of Theorem 3.3

In section 3.4 we evaluated the kernel of the operator $\mathfrak{Z}_m \mathcal{B}[S] \mathfrak{Z}_n^{-1}$ up to a scalar factor, i.e.,

$$\mathfrak{Z}_m \mathcal{B}[S] \mathfrak{Z}_n^{-1} = \sigma(m, n; S) \mathcal{Q}_{n, m}[S] \quad (3.10)$$

where $\sigma(m, n; S) \in \mathbb{C}$. We intend to prove that

$$\sigma(m, n; S) = 1 \quad \text{for all } m, n, S. \quad (3.11)$$

A priori we know the following facts about the function σ .

Lemma 3.6 a) $\sigma(m, n; S)$ is a holomorphic function in the variable S .

$$b) \quad \sigma(0, m; S) = 1 \quad ;$$

$$c) \quad \sigma(m, n; S(P_1))\sigma(k, m; S(P_2)) = \sigma(k, n; S(P_1 P_2)) \quad (3.12)$$

PROOF. a) Indeed, the both part of the equality (3.10) are holomorphic in S .

b) Indeed, the Gauss operators corresponding to elements $\text{Mor}_{\mathbf{Sp}}(V_0, V_m)$ are described at the end of Section 2.6. Due to Section 3.2, we know their images with respect to the Zak transform exactly, not up to a constant multiplicative factor.

c) We observe (see (2.3), (3.6)) that in the identities

$$\mathcal{B}[S(P_1)]\mathcal{B}[S(P_2)] = \lambda(P_1, P_2)\mathcal{B}[S(P_1 P_2)]$$

$$\mathcal{Q}_{m,n}[S(P_1)]\mathcal{Q}_{k,m}(S(P_2)) = \lambda(P_1, P_2)\mathcal{Q}_{k,n}(S(P_1 P_2))$$

the scalar factors $\lambda(P_1, P_2)$ coincide. This proves (3.12). \square

Below we easily reduce (3.11) to our lemma.

Step 1. For each idempotent $P \in \text{End}(V_n)$, the identity (3.10) implies $\sigma(n, n; S(P)) = 1$. Thus we need the description of idempotents in $\text{End}_{\mathbf{Sp}}(V_n)$.

Lemma 3.7 a) An element $P \in \text{End}_{\mathbf{Sp}}(V_n)$ is an idempotent (i.e. $PP = P$) iff $P \subset V_n \oplus V_n$ has the form $P = Y \oplus Z$, where Y is a linear subspace in the first copy of V_n and Z is a linear subspace in the second copy of V_n .

b) Let $P \in \text{End}(V_n)$ be an idempotent. Then for each $Q \in \text{End}(V_n)$, the linear relation QP is an idempotent.

The proof of a) is a straightforward verification. The statement b) is a corollary of a).

Step 2. Considering an idempotent $P \in \text{End}(V_n)$ and arbitrary $Q \in \text{End}(V_n)$, we obtain

$$\sigma(n, n; S(Q)) = \sigma(n, n; S(QP))/\sigma(n, n; S(P)) = 1/1$$

i.e.,

$$\sigma(n, n; S(Q)) = 1 \quad \text{for all } Q \in \text{End}(V_n). \quad (3.13)$$

Step 3. Let $n > m$. Now fix $R \in \text{Mor}(V_n, V_m)$ of the maximal possible rank. Let P range in $\text{End}(V_n)$. Then the set of all possible products RP is open (nondense) in $\text{Mor}(V_n, V_m)$.

By (3.12),(3.13) we have:

$$\sigma(n, m; S(RP)) = \sigma(n, n, P)\sigma(n, m; S(R)) = \sigma(n, m; S(R)).$$

Hence the expression $\sigma(n, m; S(T))$ is a constant on the open subset consisting of products RP . But $\sigma(n, m; S(R))$ is holomorphic and hence it is some constant $\sigma_{n,m}$.

For $m > n$ we repeat the same considerations with products PR with fixed $R \in \text{Mor}(V_n, V_m)$ and P ranging in $\text{End}(V_n)$.

Step 4. Now we have

$$\sigma_{0,n}\sigma_{n,m} = \sigma_{0,m}.$$

But $\sigma_{0,k} = 1$ for all k . Thus $\sigma_{m,n} = 1$. This finishes proof of the equality (3.11).

References

- [1] J. Brezin, *Harmonic analysis on nilmanifolds*, Trans. AMS **150**(1970), 611-618.
- [2] P. Cartier, *Quantum mechanical commutation relations and theta functions*, in *Algebraic groups and discontinuous subgroups*, Proc. Sympos. Pure Math. **9**, AMS, 1966, 361-383.
- [3] G. Folland, *Harmonic analysis in phase space*, Annals of Math. Studies **122**, Princeton University Press, 1989.
- [4] K. Friedrichs, *Mathematical aspects of the quantum theory of fields*, Interscience Publ., London, 1953.
- [5] L. Hörmander, *The analysis of linear partial differential operators I. Distribution theory and Fourier analysis*. Springer-Verlag, 1990.
- [6] L. Hörmander, *Symplectic classification of quadratic forms and general Mehler formulas*, Math. Z. **219**(1995), no. 3, 413-449.
- [7] R. Howe, *The oscillator semigroup*, in *The mathematical heritage of Hermann Weyl*, Proc. Sympos. Pure Math. **48**, AMS, 1988, 61-132.
- [8] A. Kirillov, A. Gvishiani, *Theorems and problems in functional analysis*, Springer-Verlag, 1982.
- [9] Krein, M. G.; Smuljan, Ju. L. *Fractional linear transformations with operator coefficients*. (Russian) Akademiya Nauk Moldavskoi SSR. Matematicheskie Issledovaniya, 2 1967 vyp. 3, 64–96.
- [10] G. Lion, M. Vergne, *The Weil representation, Maslov index and theta series*, Progr. in Math. **6**, Birkhäuser, 1980.

- [11] D. Mumford, *Tata lectures on theta I*, Progr. in Math. **28**, Birkhäuser, 1983.
- [12] D. Mumford, *Tata lectures on theta II*, Progr. in Math. **43**, Birkhäuser, 1984.
- [13] D. Mumford, *Tata lectures on theta III*, Progr. in Math. **97**, Birkhäuser, 1991.
- [14] M. Nazarov, Yu. Neretin, G. Olshanskii, *Semi-groupes engendrés par la représentation de Weil du groupe symplectique de dimension infinie*, C. R. Acad. Sci. Paris Sér. I Math. **309**(1989), no. 7, 443-446.
- [15] Yu. Neretin, *On a semigroup of operators in the boson Fock space*, Funct. Anal. Appl. **24**(1990), no. 2, 135-144.
- [16] Yu. Neretin, *Integral operators with Gaussian kernels and symmetries of canonical commutation relations*, in *Contemporary mathematical physics*, AMS Transl. Ser. 2, **175**, AMS, 1996, 97-135.
- [17] Yu. Neretin, *Categories of symmetries and infinite-dimensional groups*, London Math. Soc. Monographs, N. S., **16**, Oxford University Press, 1996.
- [18] Yu. Neretin, *Structures of boson and fermion Fock spaces in the space of symmetric functions*, Preprint 2003, available via <http://xxx.lanl.gov/abs/math-ph/0306077>.
- [19] G. Olshanskii, *The Weyl representation and the norms of Gaussian operators*, Funct. Anal. Appl. **28**(1994), no. 1, 42-54.
- [20] G. Olshanskii, *Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series*, Funct. Anal. Appl. **15**(1981), no. 4, 275-285.
- [21] J. Zak, *Finite translations in solid-state physics*, Phys. Rev. Lett. **19**(1967), 1385-1387.